

1. Examples on Lagrange's equations - II

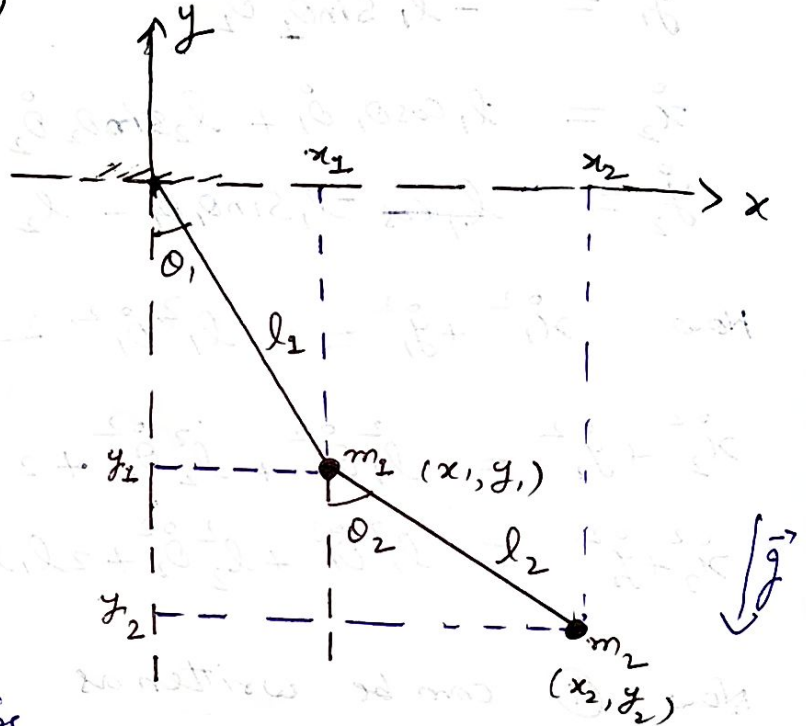
Q. Find the Lagrange's equations for a coplanar double pendulum. (see Fig)

Soln:

$l_1, l_2 \rightarrow$ lengths of strings

$\theta_1, \theta_2 \rightarrow$ angles which l_1 & l_2 make with the vertical

$m_1, m_2 \rightarrow$ masses of bobs



From Fig, positions of the bobs are given by

$$x_1 = l_1 \sin \theta_1, \quad y_1 = -l_1 \cos \theta_1 \quad \text{--- (1)}$$

$$\left. \begin{aligned} x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2 \\ y_2 &= -l_1 \cos \theta_1 - l_2 \cos \theta_2 \end{aligned} \right\} \text{--- (2)}$$

Kinetic energy of the system is given by

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \quad \text{--- (3)}$$

Potential energy is given by

$$V = m_1 g y_1 + m_2 g y_2 \quad \text{--- (4)}$$

From equations (1) & (2) time derivatives of position coordinates (x_1, y_1) (x_2, y_2) are given by

$$\dot{x}_1 = l_1 \cos \theta_1 \dot{\theta}_1, \quad \dot{y}_1 = -l_1 \sin \theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = l_1 \cos \theta_1 \dot{\theta}_1 + l_2 \cos \theta_2 \dot{\theta}_2$$

$$\dot{y}_2 = -l_1 \sin \theta_1 \dot{\theta}_1 + l_2 \sin \theta_2 \dot{\theta}_2$$

$$\text{Now } \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2 \quad \text{--- (5)}$$

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)$$

$$\text{or } \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos(\theta_1 - \theta_2)) \quad \text{--- (6)}$$

Now From Eqs (3), (5) & (6), we obtain

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] \quad \text{--- (7)}$$

Next, Potential energy V , is given by

$$V = -m_1 g l_1 \cos \theta_1 - m_2 g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2$$

$$\text{or } V = -(m_1 + m_2) g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2 \quad \text{--- (8)}$$

Now the Lagrangian $L = T - V$, is given by

$$L = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$+ (m_1 + m_2) g l_1 \cos \theta_1 + m_2 g l_2 \cos \theta_2 \quad \text{--- (9)}$$

Now the Lagrange's equations for generalized coordinates θ_1 & θ_2 are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0 \quad \text{--- (10)}$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0 \quad \text{--- (11)}$$

From Eq. (9)

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

and

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - l_1 g (m_1 + m_2) \sin \theta_1 \quad \text{--- (12)}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \quad \text{--- (13)}$$

Again For θ_2 ,

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \quad \text{--- (14)}$$

and

$$\frac{\partial L}{\partial \theta_2} = -l_2 g m_2 \sin \theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \quad \text{--- (15)}$$

From (16), (12) and (13)

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + l_1 g (m_1 + m_2) \sin \theta_1 = 0$$

$$\text{or } (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) \sin \theta_1 \\ = 0$$

→ (16)

Again From Eqs. (11), (14) & (15)

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \\ + l_2 g m_2 \sin \theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) = 0$$

or

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin \theta_2 = 0$$

Thus, the Lagrange's equations for double pendulum (17)

is ~~given~~ given by (16) & (17)

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) \sin \theta_1 = 0$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin \theta_2 = 0$$